

Carleson measures for Besov-Sobolev spaces with applications in the unit ball of \mathbf{C}^{n*}

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Abstract. This paper is devoted to give the connections between Carleson measures for Besov-Sobolev spaces $B_p^\sigma(B)$ and p -Carleson measure in the unit ball of \mathbf{C}^n . As applications, we characterize the Riemann-Stieltjes operators and multipliers acting on $B_p^\sigma(B)$ spaces by means of Carleson measures for $B_p^\sigma(B)$.

Keywords: Carleson measures, Besov- Sobolev spaces, Riemann-Stieltjes operators, Multipliers

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§1 Introduction

Let $B = \{z \in \mathbf{C}^n : |z| < 1\}$ be the unit ball of \mathbf{C}^n ($n > 1$), $S = \{z \in \mathbf{C}^n : |z| = 1\}$ be its boundary. Let dv denote the normalized Lebesgue measure of B , i.e. $v(B) = 1$, and $d\sigma$ denote the normalized rotation invariant Lebesgue measure of S satisfying $\sigma(S) = 1$. Let $d\lambda(z) = (1 - |z|^2)^{-n-1} dv(z)$ be the invariant measure on the ball.

We denote the class of all holomorphic functions in B by $H(B)$. In [3], for integer $m > 0$, and for $0 \leq \sigma < \infty$, $1 < p < \infty$, $m + \sigma > \frac{n}{p}$, the Besov-Sobolev spaces $B_p^\sigma(B)$ are defined to consist of those $f \in H(B)$ on the ball such that

$$\left\{ \sum_{k=0}^{m-1} |f^{(k)}(0)|^p + \int_B |(1 - |z|^2)^{m+\sigma} f^{(m)}(z)|^p d\lambda(z) \right\}^{\frac{1}{p}} < \infty. \quad (1.1)$$

Here $f^{(m)}$ is the m^{th} order complex derivative of f . The spaces $B_p^\sigma(B)$ are independent of m and are Banach spaces with norms given in (1.1).

For $p = 2$, these are Hilbert spaces with the usual inner product in \mathbf{C}^n . This scale of spaces includes the Dirichlet spaces $B_2(B) = B_2^0(B)$, weighted Dirichlet-type spaces with $0 < \sigma < \frac{1}{2}$, the Drury-Arveson Hardy spaces $H_n^2 = B_2^{\frac{1}{2}}(B)$, the Hardy spaces $H^2 = B_2^{\frac{n}{2}}(B)$, and the weighted Bergman spaces with $\sigma > \frac{n}{2}$ (see [14] and [22]).

For $f \in H(B)$, $z \in B$, its complex gradient and invariant gradient are defined as

$$\nabla f(z) = \nabla_z f = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right), \quad \tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0),$$

where φ_z is the Möbius transformation for $z \in B$, which satisfies $\varphi_z(0) = z$, $\varphi_z(z) = 0$ and $\varphi_z \circ \varphi_z = I$, and its radial derivative $Rf(z) = \langle \nabla f(z), \bar{z} \rangle = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(z) z_j$. In [22], the invertible

”radial” operators $R^{\alpha,t} : H(B) \rightarrow H(B)$ is denoted by

$$R^{\alpha,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)} f_k(z),$$

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provided neither $n + \alpha$ nor $n + \alpha + t$ is a negative integer, and where $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f . If the inverse of $R^{\alpha,t}$ is denoted by $R_{\alpha,t}$, then Proposition 1.14 of [22] yields

$$R^{\alpha,t} \left(\frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}},$$

$$R_{\alpha,t} \left(\frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} \right) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}},$$

for all $w \in B$. Thus for any α , $R^{\alpha,t}$ is approximately differentiation of order t .

Using the similar method of Lemma 6.3, Theorem 6.1 and Theorem 6.4 of [22], we know the definition (1.1) is equivalent to the seminorm

$$\|f\|_{B_p^\sigma(B)}^p = \int_B |R^{\alpha,m} f(z)|^p (1 - |z|^2)^{mp+\sigma p-n-1} dv(z) < \infty, \quad (1.2)$$

for integer $m > 0$, and for $0 \leq \sigma < \infty$, $1 < p < \infty$, $m + \sigma > \frac{n}{p}$, where neither $n + \alpha$ nor $n + \alpha + m$ is a negative integer.

A positive Borel measure μ on B is called a Carleson measure for $B_p^\sigma(B)$ if there is a constant $C > 0$ such that

$$\int_B |f(z)|^p d\mu(z) \leq C \|f\|_{B_p^\sigma(B)}^p, \quad \forall f \in B_p^\sigma(B).$$

For $z \in B$ and $r > 0$, we denote $E(z, r) = \{w \in B : |\varphi_z(w)| < r\}$ the pseudo-hyperbolic metric ball at z . For $\xi \in S$ and $\delta > 0$, let $Q_\delta(\xi) = \{z \in B : |1 - \langle z, \xi \rangle| < \delta\}$. For a positive Borel measure μ on B , if

$$\|\mu\|_{CM_p}^2 = \sup \left\{ \frac{\mu(Q_\delta(\xi))}{\delta^{np}}; \xi \in S, \delta > 0 \right\} < \infty,$$

we call μ a p -Carleson measure.

The study of Carleson measures for Besov-Sobolev spaces has a long history. It plays the important role in function spaces and operator theory. In one variable, various authors give their characterizations by using appropriate capacities (see [8], [17] and [19]). Recently, N. Arcozzi, R. Rochberg and E. Sawyer extend themselves earlier characterization in [1] to higher dimensions. In [2], they described the Carleson measures for $B_p^\sigma(B)$ on the unit ball in \mathbf{C}^n for $\sigma = 0$ and $1 < p < 2 + \frac{1}{n-1}$ (the difficult range $p \in [2 + \frac{1}{n-1}, \infty)$ remains open) in terms of a discrete tree condition on the associated Bergman tree. Subsequently, in [3], they considered $0 \leq \sigma \leq \frac{1}{2}$, and focused their attention on the Hilbert spaces $p = 2$ (the range $\frac{1}{2} < \sigma < \frac{n}{2}$ remains mysterious, recently, the difficult range $\frac{1}{2} < \sigma < \frac{n}{2}$ was given in [18] by a ‘‘T1 Condition’’, but only for $p = 2$). Our starting point is an attempt to get easier conditions to characterize Carleson measures for $B_p^\sigma(B)$ in the unit ball of \mathbf{C}^n for the whole values of p and σ .

In this paper, we give the connections between Carleson measures for $B_p^\sigma(B)$ and p -Carleson measure in Theorem 2.1, which seems easier to be verified than ‘‘T1 Condition’’ and the discrete tree conditions. In Theorem 2.1, we consider not only the case $\sigma = 0$ but also the case $0 < \sigma < \infty$, and these results hold for all the ranges $1 < p < \infty$. The reason why there is a difference of any small $\varepsilon > 0$ between necessary condition and sufficient condition is that p -Carleson measure is weaker than Carleson measures for $B_p^\sigma(B)$, which was mentioned in [1] for the case of the unit disc. Therefore, even in one-dimensional situation, in [6], unified necessary and sufficient condition holds only for $0 < p < q < \infty$. This paper may be regarded as an extension of [6] to the case $p = q$ in the higher dimensions. But the traditional method of one complex variable in [6] is not applicable to several complex variables. The key point in the proof of Theorem 2.1 is that duality theorem is adapted. At the same time, we apply these results to characterize Riemann-Stieltjes operators and multipliers for $B_p^\sigma(B)$ in the unit ball of \mathbf{C}^n .

The Riemann-Stieltjes operators V_φ and U_φ with the holomorphic symbol φ on B are defined as follows (see [7], [20]) :

$$V_\varphi f(z) = \int_0^1 f(tz) R\varphi(tz) \frac{dt}{t}, \quad U_\varphi f(z) = \int_0^1 \varphi(tz) Rf(tz) \frac{dt}{t}, \quad z \in B.$$

It is easy to see that the pointwise multipliers M_φ are determined by

$$M_\varphi f(z) = \varphi(z)f(z) = \varphi(0)f(0) + V_\varphi f(z) + U_\varphi f(z), \quad z \in B.$$

Of course, in the above definition f is assumed to be holomorphic in B . Clearly, $V_\varphi f = U_f \varphi$ and the Riemann-Stieltjes operator can be viewed as a generalization of the well known Cesàro operator.

Throughout this paper, C , M denote positive constants which are not necessarily the same at each appearance. The expression $A \approx B$ means that there exists a positive C such that $C^{-1}B \leq A \leq CB$.

§2 Carleson measures for Besov- Sobolev spaces

Similar to the proof of Lemma 3.2 of [13], it is easy to prove the following Lemma 2.1 and Lemma 2.2. For the convenience of readers, we give the details of the proof of Lemma 2.1.

Lemma 2.1 Let $1 < p < \infty$, μ be a positive Borel measure. Then the following statements are equivalent :

(i) The measure μ satisfies

$$\sup\{\mu(Q_\delta(\xi)); \xi \in S\} \leq C\delta^{np}.$$

(ii) For every $s > 0$,

$$\sup\left\{\int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{np+s}} d\mu(z); w \in B\right\} < \infty.$$

(iii) For some $s > 0$,

$$\sup\left\{\int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{np+s}} d\mu(z); w \in B\right\} < \infty.$$

Proof To show that (i) implies (ii). Clearly, it is sufficient to prove (ii) for $w \in B$ near to the boundary. Let J_w be the integer part of $\left(\log_2 \frac{1}{1-|w|^2}\right) - 1$. For $j = 0, 1, \dots, J_w$, consider the sets

$$\Omega_0 = \emptyset, \quad \Omega_j = \{z \in B : |1 - \langle z, w \rangle| \leq 2^j(1 - |w|^2)\}, j \geq 1$$

Thus, (ii) follows from

$$\begin{aligned}
& \int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{np+s}} d\mu(z) \\
& \leq C \sum_{j=1}^{J_w} \frac{(1 - |w|^2)^s}{2^{(np+s)(j-1)} (1 - |w|^2)^{np+s}} \mu(\Omega_j \setminus \Omega_{j-1}) \\
& \quad + \int_{B \setminus \Omega_{J_w}} \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{np+s}} d\mu(z) \\
& \leq C \sum_{j=1}^{J_w} \frac{1}{2^{(np+s)(j-1)} (1 - |w|^2)^{np}} \mu(\Omega_j) + C \mu(B \setminus \Omega_{J_w}) \\
& \leq C \sum_{j=1}^{J_w} \frac{1}{2^{(np+s)(j-1)} (1 - |w|^2)^{np}} \left(2^j (1 - |w|^2) \right)^{np} + C \\
& \leq C \sum_{j=0}^{\infty} \frac{1}{2^{sj}} < \infty.
\end{aligned}$$

That (ii) implies (iii) is trivial. To obtain (i) from (iii), note that for $0 < \delta < 1$ and $z \in Q_\delta(\xi)$ we have $|1 - \langle z, (1 - \delta)\xi \rangle| \approx \delta$. Therefore,

$$\begin{aligned}
\mu(Q_\delta(\xi)) & \leq C \delta^{np} \int_{Q_\delta(\xi)} \frac{(1 - |(1 - \delta)\xi|^2)^s}{|1 - \langle z, (1 - \delta)\xi \rangle|^{np+s}} d\mu(z) \\
& \leq C \delta^{np}.
\end{aligned}$$

Lemma 2.2 Let $1 < p < \infty$, $\varepsilon > 0$ and μ be a positive Borel measure. Then the following statements are equivalent :

(i) The measure μ satisfies

$$\sup\{\mu(Q_\delta(\xi)); \xi \in S\} \leq C \log^{1-p-\varepsilon} \frac{2}{\delta}.$$

(ii) For every $s > 0$,

$$\sup\{\log^{p-1+\varepsilon} \frac{2}{1 - |w|^2} \int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^s} d\mu(z); w \in B\} < \infty.$$

(iii) For some $s > 0$,

$$\sup\{\log^{p-1+\varepsilon} \frac{2}{1 - |w|^2} \int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^s} d\mu(z); w \in B\} < \infty.$$

Lemma 2.3 Suppose integer $m > 0$, $1 < p < \infty$, $\varepsilon > 0$, neither $n + \alpha$ nor $n + \alpha + m$ is a negative integer, μ be a positive Borel measure such that

$$\sup\left\{ \frac{\mu(Q_\delta(\xi))}{\log^{1-p-\varepsilon} \frac{2}{\delta}}; \xi \in S, \delta > 0 \right\} < \infty.$$

Then, for $M > mp$,

$$\int_B \left(\int_B |R^{\alpha, m} f(w)| \frac{(1 - |w|^2)^M}{|1 - \langle z, w \rangle|^{n+M+1-m}} dv(w) \right)^p d\mu(z) \leq C \|f\|_{B_p^0(B)}^p.$$

Proof Let

$$I = \left(\int_B \left(\int_B |R^{\alpha,m} f(w)| \frac{(1-|w|^2)^M}{|1-\langle z, w \rangle|^{n+M+1-m}} dv(w) \right)^p d\mu(z) \right)^{\frac{1}{p}}.$$

Let $\|\cdot\|_{L_p}$ denote the usual norm on $L^p(B, d\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. By duality,

$$I = \sup_{\|\psi\|_{L_q}=1} \left\{ \int_B \int_B \frac{|R^{\alpha,m} f(w)| (1-|w|^2)^M}{|1-\langle z, w \rangle|^{n+M+1-m}} dv(w) |\psi(z)| d\mu(z) \right\}.$$

By Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned} I &\leq \sup_{\|\psi\|_{L_q}=1} \left(\int_B \int_B \frac{|R^{\alpha,m} f(w)|^p (1-|w|^2)^M \log^{p-1+\varepsilon} \frac{2}{1-|w|^2}}{|1-\langle z, w \rangle|^{M-mp+n+1}} dv(w) d\mu(z) \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_B \int_B \frac{|\psi(z)|^q (1-|w|^2)^M}{|1-\langle z, w \rangle|^{n+1+M} \log^{1+\frac{q}{p}\varepsilon} \frac{2}{1-|w|^2}} dv(w) d\mu(z) \right)^{\frac{1}{q}} \\ &\leq C \sup_{\|\psi\|_{L_q}=1} \left(\int_B \int_B \frac{(1-|w|^2)^{M-mp} \log^{p-1+\varepsilon} \frac{2}{1-|w|^2}}{|1-\langle z, w \rangle|^{M-mp}} d\mu(z) |R^{\alpha,m} f(w)|^p (1-|w|^2)^{mp-n-1} dv(w) \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_B \int_B \frac{(1-|w|^2)^M}{|1-\langle z, w \rangle|^{n+1+M} \log^{1+\frac{q}{p}\varepsilon} \frac{2}{1-|w|^2}} dv(w) |\psi(z)|^q d\mu(z) \right)^{\frac{1}{q}}. \end{aligned}$$

Similar to the proof of Lemma 3.4 in [11], it is clear that the inner integral of the last line above is bounded. And noting $M > mp$, by Lemma 2.2 we can get

$$\begin{aligned} I &\leq C \sup_{\|\psi\|_{L_q}=1} \left(\int_B |R^{\alpha,m} f(w)|^p (1-|w|^2)^{mp-n-1} dv(w) \right)^{\frac{1}{p}} \times \left(\int_B |\psi(z)|^q d\mu(z) \right)^{\frac{1}{q}} \\ &\leq C \left(\int_B |R^{\alpha,m} f(w)|^p (1-|w|^2)^{mp-n-1} dv(w) \right)^{\frac{1}{p}} \\ &\leq C \|f\|_{B_p^0(B)}. \end{aligned}$$

Theorem 2.1 Suppose integer $m > 0$, $0 \leq \sigma < \infty$, $1 < p < \infty$, $m + \sigma > \frac{n}{p}$, neither $n + \alpha$ nor $n + \alpha + m$ is a negative integer. Let μ be a positive Borel measure in B .

(i) For $\sigma > 0$, if μ is a Carleson measure for $B_p^\sigma(B)$, then μ is a $\frac{\sigma p}{n}$ -Carleson measure, conversely, if $\varepsilon > 0$ and μ is a $\frac{\sigma p + \varepsilon}{n}$ -Carleson measure, then μ is a Carleson measure for $B_p^\sigma(B)$.

(ii) For $\sigma = 0$, if μ is a Carleson measure for $B_p^0(B)$, then μ satisfy

$$\sup \left\{ \frac{\mu(Q_\delta(\xi))}{\log^{1-p} \frac{2}{\delta}}; \xi \in S, \delta > 0 \right\} < \infty,$$

conversely, if $\varepsilon > 0$ and μ satisfy

$$\sup \left\{ \frac{\mu(Q_\delta(\xi))}{\log^{1-p-\varepsilon} \frac{2}{\delta}}; \xi \in S, \delta > 0 \right\} < \infty,$$

then μ is a Carleson measure for $B_p^0(B)$.

Proof

(i) For $\sigma > 0$, suppose first that μ is a Carleson measure for $B_p^\sigma(B)$, then

$$\|f\|_{L^p(d\mu)} \leq C\|f\|_{B_p^\sigma(B)}. \quad (2.1)$$

We can find a constant $s > 0$ such that $\sigma + \frac{s}{p} - n - 1 = \alpha + N$ for some positive integer N . Applying (2.1) to the test functions

$$f_w(z) = \frac{(1 - |w|^2)^{\frac{s}{p}}}{(1 - \langle z, w \rangle)^{\sigma + \frac{s}{p}}}, \quad w \in B,$$

by Lemma 2.18 of [22], we can get

$$\begin{aligned} \int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{\sigma p + s}} d\mu(z) &\leq C \int_B |R^{\alpha, m} f_w(z)|^p (1 - |z|^2)^{p(\sigma + m) - n - 1} dv(z) \\ &\leq C \int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{\sigma p + mp + s}} (1 - |z|^2)^{p(\sigma + m) - n - 1} dv(z) \\ &= C \int_B \frac{(1 - |w|^2)^s (1 - |z|^2)^{p(\sigma + m) - n - 1}}{|1 - \langle z, w \rangle|^{(n+1) + [p(\sigma + m) - n - 1] + s}} dv(z) \\ &\leq C(1 - |w|^2)^s (1 - |w|^2)^{-s} \\ &\leq C. \end{aligned} \quad (2.2)$$

where Proposition 1.4.10 of [16] is used. Taking sup of (2.2) and by Lemma 2.1, we know that μ is a $\frac{\sigma p}{n}$ -Carleson measure.

On the other hand, suppose μ is a $\frac{\sigma p + \varepsilon}{n}$ -Carleson measure, we need to prove μ is a Carleson measure for $B_p^\sigma(B)$.

Fix a sufficiently large positive integer K and let $M = \alpha + K$. Then

$$R^{\alpha, m} f(z) = C_M \int_B R^{\alpha, m} f(w) \frac{(1 - |w|^2)^M}{(1 - \langle z, w \rangle)^{n+1+M}} dv(w).$$

Acting on the above equation by the inverse operator $R_{\alpha, m}$,

$$f(z) = C_M R_{\alpha, m} \int_B R^{\alpha, m} f(w) \frac{(1 - |w|^2)^M}{(1 - \langle z, w \rangle)^{n+1+M}} dv(w).$$

By Lemma 2.18 of [22], there exists a polynomial $P(z, w)$ such that

$$f(z) = C_M \int_B \frac{P(z, w) R^{\alpha, m} f(w) (1 - |w|^2)^M}{(1 - \langle z, w \rangle)^{n+1+M-m}} dv(w),$$

and consequently, we can get

$$|f(z)| \leq C \int_B |R^{\alpha, m} f(w)| \frac{(1 - |w|^2)^M}{|1 - \langle z, w \rangle|^{n+M+1-m}} dv(w). \quad (2.3)$$

By (2.3) and a process similar to the proof of Lemma 2.3, we have

$$\begin{aligned}
& \left(\int_B |f(z)|^p d\mu(z) \right)^{\frac{1}{p}} \\
& \leq C \left(\int_B \left(\int_B |R^{\alpha,m} f(w)| \frac{(1-|w|^2)^M}{|1-\langle z, w \rangle|^{n+M+1-m}} dv(w) \right)^p d\mu(z) \right)^{\frac{1}{p}} \\
& = C \sup_{\|\psi\|_{L_q}=1} \left\{ \int_B \int_B \frac{|R^{\alpha,m} f(w)| (1-|w|^2)^M}{|1-\langle z, w \rangle|^{n+M+1-m}} dv(w) |\psi(z)| d\mu(z) \right\} \\
& \leq C \left(\int_B \int_B \frac{|R^{\alpha,m} f(w)|^p (1-|w|^2)^M}{|1-\langle z, w \rangle|^{M-mp+n+1+\varepsilon}} dv(w) d\mu(z) \right)^{\frac{1}{p}} \\
& \times \sup_{\|\psi\|_{L_q}=1} \left(\int_B \int_B \frac{|\psi(z)|^q (1-|w|^2)^M}{|1-\langle z, w \rangle|^{n+1+M-\frac{q}{p}\varepsilon}} dv(w) d\mu(z) \right)^{\frac{1}{q}} \\
& \leq C \left(\int_B \left(\int_B \frac{(1-|w|^2)^{M-[(\sigma+m)p-n-1]}}{|1-\langle z, w \rangle|^{M-[(\sigma+m)p-n-1]+(\sigma p+\varepsilon)}} d\mu(z) \right) |R^{\alpha,m} f(w)|^p (1-|w|^2)^{(\sigma+m)p-n-1} dv(w) \right)^{\frac{1}{p}} \\
& \times \sup_{\|\psi\|_{L_q}=1} \left(\int_B \left(\int_B \frac{(1-|w|^2)^M}{|1-\langle z, w \rangle|^{n+1+M-\frac{q}{p}\varepsilon}} dv(w) \right) |\psi(z)|^q d\mu(z) \right)^{\frac{1}{q}}. \quad (2.4)
\end{aligned}$$

Applying Lemma 2.1 and Proposition 1.4.10 of [16] to the two inner integrals in the end of (2.4) respectively, we know

$$\begin{aligned}
(2.4) & \leq C \left(\int_B |R^{\alpha,m} f(w)|^p (1-|w|^2)^{(\sigma+m)p-n-1} dv(w) \right)^{\frac{1}{p}} \times \sup_{\|\psi\|_{L_q}=1} \left(\int_B |\psi(z)|^q d\mu(z) \right)^{\frac{1}{q}} \\
& \leq C \left(\int_B |R^{\alpha,m} f(w)|^p (1-|w|^2)^{(\sigma+m)p-n-1} dv(w) \right)^{\frac{1}{p}} \\
& \leq C \|f\|_{B_p^\sigma(B)}.
\end{aligned}$$

this implies μ is a Carleson measure for $B_p^\sigma(B)$.

(ii) For $\sigma = 0$, suppose μ is a Carleson measure for $B_p^0(B)$, we need to prove

$$\sup \left\{ \frac{\mu(Q_\delta(\xi))}{\log^{1-p} \frac{2}{\delta}}; \xi \in S, \delta > 0 \right\} < \infty.$$

For any $\xi \in S$, and $0 < \delta < 1$, we consider the functions

$$f_{\xi,\delta}(z) = \log^{-\frac{1}{p}} \frac{2}{\delta} \log \frac{2}{1-\langle z, (1-\delta)\xi \rangle}.$$

Since $\log \frac{2}{1-\langle z, w \rangle} = \log 2 + \sum_{k=1}^{\infty} k^{-1} \langle z, w \rangle^k$, and by induction

$$R^{\alpha,m} f(z) = \sum_{k=0}^{\infty} \frac{(n+\alpha+1+k) \cdots (n+\alpha+m+k)}{(n+\alpha+1) \cdots (n+\alpha+m)} f_k(z), \quad m \in N,$$

we can get

$$\left| R^{\alpha, m} \log \frac{2}{1 - \langle z, w \rangle} \right| \approx \left| \sum_{k=0}^{\infty} k^{m-1} \langle z, w \rangle^k \right| \approx |(1 - \langle z, w \rangle)^{-m}|,$$

for all $w \in B$. The last formula is due to $(1 - \langle z, w \rangle)^{-m} = \sum_{k=0}^{\infty} \frac{\Gamma(k+m)}{k! \Gamma(m)} \langle z, w \rangle^k$. Thus, using Proposition 1.4.10 of [16] again, we know

$$\begin{aligned} \int_{Q_\delta(\xi)} |f_{\xi, \delta}(z)|^p d\mu(z) &\leq \int_B |f_{\xi, \delta}(z)|^p d\mu(z) \\ &\leq C \int_B |R^{\alpha, m} f_{\xi, \delta}(z)|^p (1 - |z|^2)^{mp-n-1} dv(z) \\ &\leq C \int_B \frac{(1 - |z|^2)^{mp-n-1}}{|1 - \langle z, (1 - \delta)\xi \rangle|^{mp}} \log^{-1} \frac{2}{\delta} dv(z) \\ &= C \log^{-1} \frac{2}{\delta} \int_B \frac{(1 - |z|^2)^{mp-n-1}}{|1 - \langle z, (1 - \delta)\xi \rangle|^{(n+1)+(mp-n-1)}} dv(z) \\ &\leq C \log^{-1} \frac{2}{\delta} \log \frac{2}{\delta} \\ &\leq C, \end{aligned}$$

where the condition $m + \sigma > \frac{n}{p}$ is applied. By Lemma 2.6 of [12], we have $|f_{\xi, \delta}(z)| \approx \log^{1-\frac{1}{p}} \frac{2}{\delta}$ for $z \in Q_\delta(\xi)$. Consequently,

$$\sup \left\{ \frac{\mu(Q_\delta(\xi))}{\log^{1-p} \frac{2}{\delta}}; \xi \in S, \delta > 0 \right\} < \infty.$$

On the other hand, suppose

$$\sup \left\{ \frac{\mu(Q_\delta(\xi))}{\log^{1-p-\varepsilon} \frac{2}{\delta}}; \xi \in S, \delta > 0 \right\} < \infty,$$

using (2.3) provided $M > mp$ large enough and by Lemma 2.3, we have

$$\begin{aligned} \int_B |f(z)|^p d\mu(z) &\leq C \int_B \left(\int_B |R^{\alpha, m} f(w)| \frac{(1 - |w|^2)^M}{|1 - \langle z, w \rangle|^{n+M+1-m}} dv(w) \right)^p d\mu(z) \\ &\leq C \|f\|_{B_p^0(B)}^p, \end{aligned}$$

this implies μ is a Carleson measure for $B_p^0(B)$.

Remark 2.1 Theorem 2.1 is an extension of Theorem 1 in [6] to the higher dimensions. Since p -Carleson measure is weaker than Carleson measures for $B_p^\sigma(B)$ for the case $p = q$, it is natural that there exists a difference of an arbitrarily small $\varepsilon > 0$ between the necessity and the sufficiency. However, we also note that such necessary and sufficient conditions are unified, i.e. $\varepsilon = 0$ for Hardy spaces H^p and the weighted Bergman spaces A_α^p in the unit disk of \mathbf{C} (see [4], [5] and [10]), and the weighted Bergman spaces A_α^p in the unit ball of \mathbf{C}^n (see [22]) in the case $p = q$.

§3 Riemann-Stieltjes operators and multipliers for $B_p^\sigma(B)$

In the following, we will apply these results to characterize the Riemann-Stieltjes operators and multipliers for $B_p^\sigma(B)$ in the unit ball of \mathbf{C}^n .

Theorem 3.1 Suppose that $\varphi \in H(B)$, $1 < p < \infty$, $0 \leq \sigma < \infty$, $m = 1$, $1 + \sigma > \frac{n}{p}$. Then $U_\varphi : B_p^\sigma(B) \rightarrow B_p^\sigma(B)$ is bounded if and only if $\|\varphi\|_{H^\infty} < \infty$.

Proof Noting that $|R^{\alpha,1}f(z)| \approx |Rf(z)|$, we can work with the radial derivative $Rf(z)$. If $\|\varphi\|_{H^\infty} < \infty$, then

$$\begin{aligned} \int_B |R(U_\varphi f)(z)|^p (1-|z|^2)^{p+\sigma p-n-1} dv(z) &= \int_B |\varphi(z)|^p |Rf(z)|^p (1-|z|^2)^{p+\sigma p-n-1} dv(z) \\ &\leq \|\varphi\|_{H^\infty}^p \|f\|_{B_p^\sigma(B)}^p. \end{aligned}$$

So, $U_\varphi : B_p^\sigma(B) \rightarrow B_p^\sigma(B)$ is bounded.

Conversely, suppose $U_\varphi : B_p^\sigma(B) \rightarrow B_p^\sigma(B)$ is bounded. For each $w \in B$ near to the boundary with $|w| > \frac{2}{3}$. Choosing $f_w(z) = \frac{(1-|w|^2)}{(1-\langle z, w \rangle)^{1+\sigma}}$. By Proposition 1.4.10 of [16], we have

$$\begin{aligned} \int_B |Rf_w(z)|^p (1-|z|^2)^{p+\sigma p-n-1} dv(z) &= \int_B \frac{(1-|w|^2)^p |\langle z, w \rangle|^p (1-|z|^2)^{p+\sigma p-n-1}}{|1-\langle z, w \rangle|^{(2+\sigma)p}} dv(z) \\ &\leq (1-|w|^2)^p \int_B \frac{(1-|z|^2)^{p+\sigma p-n-1}}{|1-\langle z, w \rangle|^{(2+\sigma)p}} dv(z) \\ &\leq C(1-|w|^2)^p (1-|w|^2)^{-p} \\ &\leq C, \end{aligned} \tag{3.1}$$

this implies $\sup_{w \in B} \|f_w\|_{B_p^\sigma(B)} \leq C$. It is well known that

$$v(E(w, \frac{1}{2})) \approx (1-|w|^2)^{n+1}, \quad 1-|w|^2 \approx 1-|z|^2 \approx |1-\langle z, w \rangle| \quad \text{for } z \in E(w, \frac{1}{2}).$$

Also note that for $z \in E(w, \frac{1}{2})$, we have

$$1-|\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z, w \rangle|^2} > \frac{3}{4}.$$

Thus

$$1-|\langle z, w \rangle| \leq |1-\langle z, w \rangle| < \frac{2}{\sqrt{3}}(1-|w|^2)^{\frac{1}{2}}(1-|z|^2)^{\frac{1}{2}} \leq \frac{2}{\sqrt{3}}(1-|w|^2)^{\frac{1}{2}} < \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{5}}{3} = \frac{2\sqrt{15}}{9},$$

this implies $|\langle z, w \rangle| > 1 - \frac{2\sqrt{15}}{9}$. By the \mathcal{M} -subharmonicity of $|\varphi(w)|^p$, we have

$$\begin{aligned} |\varphi(w)|^p &\leq C \frac{1}{v(E(w, \frac{1}{2}))} \int_{E(w, \frac{1}{2})} |\varphi(z)|^p dv(z) \\ &\leq C \frac{1}{(1-|w|^2)^{n+1}} \int_{E(w, \frac{1}{2})} |\varphi(z)|^p dv(z) \\ &\leq C \int_{E(w, \frac{1}{2})} \frac{(1-|w|^2)^p}{|1-\langle z, w \rangle|^{(2+\sigma)p}} |\varphi(z)|^p (1-|z|^2)^{p+\sigma p-n-1} dv(z) \\ &\leq C \int_{E(w, \frac{1}{2})} \frac{|\langle z, w \rangle|^p (1-|w|^2)^p}{|1-\langle z, w \rangle|^{(2+\sigma)p}} |\varphi(z)|^p (1-|z|^2)^{p+\sigma p-n-1} dv(z) \\ &\leq C \int_B |\varphi(z)|^p |Rf_w(z)|^p (1-|z|^2)^{p+\sigma p-n-1} dv(z) \\ &\leq C \|U_\varphi(f_w)\|_{B_p^\sigma(B)}^p \leq C \|U_\varphi\|^p \|f_w\|_{B_p^\sigma(B)}^p \leq C, \end{aligned}$$

and consequently, $|\varphi(w)| \leq C$ for $|w| > \frac{2}{3}$. By maximum modulus principle, we have $|\varphi(w)| \leq C$ for $w \in B$. Thus $\varphi \in H^\infty$.

Theorem 3.2 Suppose that $\varphi \in H(B)$, $1 < p < \infty$, $0 \leq \sigma < \infty$, $m = 1$, $1 + \sigma > \frac{n}{p}$. Then the following conditions are equivalent:

- (i) $V_\varphi : B_p^\sigma(B) \rightarrow B_p^\sigma(B)$ is bounded.
- (ii) The positive Borel measure μ_φ in B defined by

$$d\mu_\varphi(z) = |R\varphi(z)|^p(1 - |z|^2)^{p+\sigma p-n-1}dv(z)$$

is a Carleson measure for $B_p^\sigma(B)$.

Proof Note that $R(V_\varphi f)(z) = f(z)R\varphi(z)$. Suppose $V_\varphi : B_p^\sigma(B) \rightarrow B_p^\sigma(B)$ is bounded. Then

$$\begin{aligned} \int_B |f(z)|^p d\mu_\varphi(z) &= \int_B |f(z)|^p |R\varphi(z)|^p (1 - |z|^2)^{p+\sigma p-n-1} dv(z) \\ &= \int_B |R(V_\varphi f)(z)|^p (1 - |z|^2)^{p+\sigma p-n-1} dv(z) \\ &\leq \|V_\varphi f\|_{B_p^\sigma(B)}^p \leq \|V_\varphi\|^p \|f\|_{B_p^\sigma(B)}^p \leq C \|f\|_{B_p^\sigma(B)}^p. \end{aligned}$$

So, $d\mu_\varphi$ is a Carleson measure for $B_p^\sigma(B)$.

Conversely, suppose $d\mu_\varphi(z) = |R\varphi(z)|^p(1 - |z|^2)^{p+\sigma p-n-1}dv(z)$ is a Carleson measure for $B_p^\sigma(B)$.

$$\begin{aligned} \|V_\varphi f\|_{B_p^\sigma(B)}^p &= \int_B |R(V_\varphi f)(z)|^p (1 - |z|^2)^{p+\sigma p-n-1} dv(z) \\ &= \int_B |f(z)|^p |R\varphi(z)|^p (1 - |z|^2)^{p+\sigma p-n-1} dv(z) \\ &= \int_B |f(z)|^p d\mu_\varphi(z) \\ &\leq C \|f\|_{B_p^\sigma(B)}^p, \end{aligned}$$

this implies $V_\varphi : B_p^\sigma(B) \rightarrow B_p^\sigma(B)$ is bounded.

Corollary 3.1 Suppose that $\varphi \in H(B)$, $1 < p < \infty$, $0 \leq \sigma < \infty$, $m = 1$, $1 + \sigma > \frac{n}{p}$. Then the following conditions are equivalent:

- (i) $M_\varphi : B_p^\sigma(B) \rightarrow B_p^\sigma(B)$ is bounded.
- (ii) $\varphi \in H^\infty$ and the positive Borel measure μ_φ in B defined by

$$d\mu_\varphi(z) = |R\varphi(z)|^p(1 - |z|^2)^{p+\sigma p-n-1}dv(z)$$

is a Carleson measure for $B_p^\sigma(B)$.

Proof The implication "(ii) \Rightarrow (i)" follows from Theorem 3.1 and Theorem 3.2, using the fact that $R(M_\varphi f)(z) = R(V_\varphi f)(z) + R(U_\varphi f)(z)$.

"(i) \Rightarrow (ii)".

Suppose $M_\varphi : B_p^\sigma(B) \rightarrow B_p^\sigma(B)$ is bounded. At first, similar to the proof of Theorem 3.1, we will prove that $\varphi \in H^\infty$. For each $w \in B$ near to the boundary with $|w| > \frac{2}{3}$, set

$$\widetilde{f}_w(z) = \frac{(1 - |w|^2)}{(1 - \langle z, w \rangle)^{1+\sigma}} - (1 - |w|^2)^{-\sigma}, \quad z \in B.$$

Since $R\widetilde{f_w}(z) = Rf_w(z)$, (3.1) implies that $\sup_{w \in B} \|\widetilde{f_w}\|_{B_p^\sigma(B)} \leq C$. Noting that $\widetilde{f_w}(w) = 0$, by the \mathcal{M} -subharmonicity of $|R(M_\varphi \widetilde{f_w})(w)|^p$, we have

$$\begin{aligned}
|\varphi(w)|^p &\leq C|w|^{2p}(1-|w|^2)^{-(1+\sigma)p}|\varphi(w)|^p(1-|w|^2)^{p+\sigma p} \\
&= C|R\widetilde{f_w}(w)|^p|\varphi(w)|^p(1-|w|^2)^{p+\sigma p} \\
&= C|R\widetilde{f_w}(w)\varphi(w) + \widetilde{f_w}(w)R\varphi(w)|^p(1-|w|^2)^{p+\sigma p} \\
&= C|R(M_\varphi \widetilde{f_w})(w)|^p(1-|w|^2)^{p+\sigma p} \\
&\leq C \frac{(1-|w|^2)^{p+\sigma p}}{v(E(w, \frac{1}{2}))} \int_{E(w, \frac{1}{2})} |R(M_\varphi \widetilde{f_w})(z)|^p dv(z) \\
&\leq C \int_B |R(M_\varphi \widetilde{f_w})(z)|^p (1-|z|^2)^{p+\sigma p-n-1} dv(z) \\
&\leq C \|M_\varphi \widetilde{f_w}\|_{B_p^\sigma(B)}^p \leq C \|M_\varphi\|^p \|\widetilde{f_w}\|_{B_p^\sigma(B)}^p \leq C,
\end{aligned}$$

and consequently, $|\varphi(w)| \leq C$ for $|w| > \frac{2}{3}$. By maximum modulus principle, we have $|\varphi(w)| \leq C$ for $w \in B$. Thus $\varphi \in H^\infty$.

Remark 3.1 As to the Riemann-Stieltjes operators and multipliers on Besov-Sobolev spaces, in the case of one complex variable, there are a lot of results, see [6], [8], [17], [19]. In the case of several complex variables, we can find the research has been developing, see [2], [3], [11], [15]. Such question on other spaces was studied in [7], [9], [12], [13], [20], [21].

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